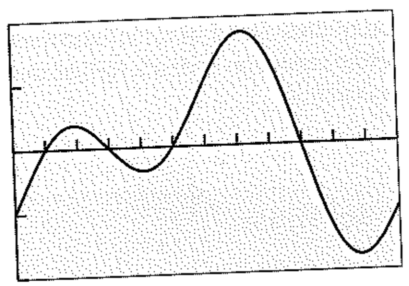
Solving Trigonometric Equations

New identities always provide new tools for solving trigonometric equations algebraically. Under the right conditions, they even lead to exact solutions. We assert again that we are not presenting these algebraic solutions for their practical value (as the calculator solutions are certainly sufficient for most applications and unquestionably much quicker to obtain), but rather as ways to observe the behavior of the trigonometric functions and their interwoven tapestry of identities.



$[0, 2\pi]$ by $[-2, 2]$

FIGURE 5.11 The function $y = \sin 2x - \cos x$ for $0 \le x \le 2\pi$. The scale on the x-axis shows intervals of length $\pi/6$. This graph supports the solution found algebraically in Example 4.

EXAMPLE 4 Using a Double-Angle Identity

Solve algebraically in the interval $[0, 2\pi)$: $\sin 2x = \cos x$.

SOLUTION

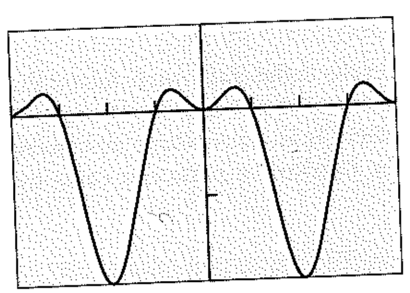
$$\sin 2x = \cos x$$
$$2 \sin x \cos x = \cos x$$
$$2 \sin x \cos x - \cos x = 0$$
$$\cos x\,(2 \sin x - 1) = 0$$
$$\cos x = 0 \quad \text{or} \quad 2 \sin x - 1 = 0$$
$$\cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$$

The two solutions of $\cos x = 0$ are $x = \pi/2$ and $x = 3\pi/2$. The two solutions of $\sin x = 1/2$ are $x = \pi/6$ and $x = 5\pi/6$. Therefore, the solutions of $\sin 2x = \cos x$ are

$$\frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{3\pi}{2}.$$

We can **support** this result **graphically** by verifying the four x-intercepts of the function $y = \sin 2x - \cos x$ in the interval $[0, 2\pi)$ (Figure 5.11).

Now try Exercise 23.



$[-2\pi, 2\pi]$ by $[-2, 1]$

FIGURE 5.12 The graph of $y = \sin^2 x - 2 \sin^2 (x/2)$ suggests that $\sin^2 x = 2 \sin^2 (x/2)$ has three solutions in $[0, 2\pi)$. (Example 5)

EXAMPLE 5 Using Half-Angle Identities

Solve $\sin^2 x = 2 \sin^2 (x/2)$.

SOLUTION The graph of $y = \sin^2 x - 2 \sin^2 (x/2)$ in Figure 5.12 suggests that this function is periodic with period 2π and that the equation $\sin^2 x = 2 \sin^2 (x/2)$ has three solutions in $[0, 2\pi)$.

Solve Algebraically

$$\sin^2 x = 2 \sin^2 \frac{x}{2}$$
$$\sin^2 x = 2\left(\frac{1 - \cos x}{2}\right) \qquad \text{Half-angle identity}$$
$$1 - \cos^2 x = 1 - \cos x \qquad \text{Convert to all cosines.}$$
$$\cos x - \cos^2 x = 0$$

UR NEXT!

continued

$$\cos x\,(1 - \cos x) = 0$$
$$\cos x = 0 \quad \text{or} \quad \cos x = 1$$
$$x = \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} \quad \text{or} \quad 0$$

The rest of the solutions are obtained by periodicity:

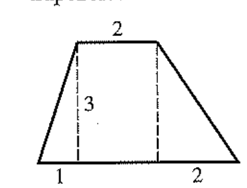
$$x = 2n\pi, \quad x = \frac{\pi}{2} + 2n\pi, \quad x = \frac{3\pi}{2} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Now try Exercise 43.

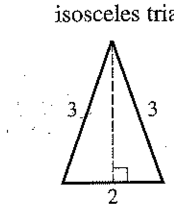
QUICK REVIEW 5.4 *(For help, go to Section 5.1.)*

In Exercises 1–8, find the general solution of the equation.

1. $\tan x - 1 = 0$
2. $\tan x + 1 = 0$
3. $(\cos x)(1 - \sin x) = 0$
4. $(\sin x)(1 + \cos x) = 0$
5. $\sin x + \cos x = 0$
6. $\sin x - \cos x = 0$
7. $(2 \sin x - 1)(2 \cos x + 1) = 0$
8. $(\sin x + 1)(2 \cos x - \sqrt{2}) = 0$
9. Find the area of the trapezoid.



10. Find the height of the isosceles triangle.



SECTION 5.4 EXERCISES

In Exercises 1–4, use the appropriate sum or difference identity to prove the double-angle identity.

1. $\cos 2u = \cos^2 u - \sin^2 u$
2. $\cos 2u = 2 \cos^2 u - 1$
3. $\cos 2u = 1 - 2 \sin^2 u$
4. $\tan 2u = \dfrac{2 \tan u}{1 - \tan^2 u}$

In Exercises 5–10, find all solutions to the equation in the interval $[0, 2\pi)$.

5. $\sin 2x = 2 \sin x$
6. $\sin 2x = \sin x$
7. $\cos 2x = \sin x$
8. $\cos 2x = \cos x$
9. $\sin 2x - \tan x = 0$
10. $\cos^2 x + \cos x = \cos 2x$

In Exercises 11–14, write the expression as one involving only $\sin \theta$ and $\cos \theta$.

11. $\sin 2\theta + \cos \theta$
12. $\sin 2\theta + \cos 2\theta$
13. $\sin 2\theta + \cos 3\theta$
14. $\sin 3\theta + \cos 2\theta$

In Exercises 15–22, prove the identity.

15. $\sin 4x = 2 \sin 2x \cos 2x$
16. $\cos 6x = 2 \cos^2 3x - 1$
17. $2 \csc 2x = \csc^2 x \tan x$
18. $2 \cot 2x = \cot x - \tan x$
19. $\sin 3x = (\sin x)(4 \cos^2 x - 1)$
20. $\sin 3x = (\sin x)(3 - 4 \sin^2 x)$
21. $\cos 4x = 1 - 8 \sin^2 x \cos^2 x$
22. $\sin 4x = (4 \sin x \cos x)(2 \cos^2 x - 1)$

In Exercises 23–30, solve algebraically for exact solutions in the interval $[0, 2\pi)$. Use your grapher only to support your algebraic work.

23. $\cos 2x + \cos x = 0$
24. $\cos 2x + \sin x = 0$
25. $\cos x + \cos 3x = 0$
26. $\sin x + \sin 3x = 0$
27. $\sin 2x + \sin 4x = 0$
28. $\cos 2x + \cos 4x = 0$
29. $\sin 2x - \cos 3x = 0$
30. $\sin 3x + \cos 2x = 0$

In Exercises 31–36, use half-angle identities to find an exact value without a calculator.

31. $\sin 15°$
32. $\tan 195°$
33. $\cos 75°$
34. $\sin (5\pi/12)$
35. $\tan (7\pi/12)$
36. $\cos (\pi/8)$